MAT 402 S26
PROBLEM SET 4

I. Let $A : \mathbb{R} \to M_{n\times n}(\mathbb{R})$ be a curve in the vector space of $n \times n$ matrices with real entries. Suppose

$$x_i = (x_i^1, ..., x_i^n) : I \to \mathbb{R}^n, \quad i = 1, ..., n$$

are n solution to the linear system of differential equations

$$\dot{x} = A(t)x.$$

Define the *Wronskian* of $x_1, ..., x_n$ to be **the determinant** of the curve of $n \times n$ matrices

$$\mathbb{R} \ni t \mapsto X(t) := \begin{pmatrix} x_1(t) \\ . \\ . \\ . \\ x_n(t) \end{pmatrix} \in M_{n\times n}(\mathbb{R})$$

whose k^{th} row is $x_k(t)$:

$$W[x](t) := \det X(t).$$

Show that

$$W[x](t) = W[x](0) \exp\left(\int_0^t \text{Trace}(A(s))ds\right).$$

II. Let $f = (f^1, ..., f^m) : \mathbb{R}^m \to \mathbb{R}^m$ be a vector valued function that vanishes at a point $x_o \in \mathbb{R}^m$, i.e., x_o is an equilibrium point of f.

i. The equilibrium point x_o of f is said to be *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in B_\delta(x_o)$ the unique solution $x(t) = (x^1(t), ..., x^n(t))$ of the initial value problem

$$\dot{x}^i = f^i(x^1, ..., x^n), \quad i = 1, ..., n, \qquad x(0) = y$$

is defined for all $t \geq 0$ and satisfies $x(t) \in B_\varepsilon(x_o)$ for all $t > 0$.

ii. The equilibrium point x_o of f is said to be *asymptotically stable* if x_o is stable and moreover

$$\lim_{t\to\infty} x(t) = x_o.$$

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function such that $\nabla H(P_o, Q_o) = 0$ for some $(P_o, Q_o) \in \mathbb{R}^{2n}$.

a. Show that (P_o, Q_o) is an equilibrium point of the Hamiltonian system determined by H.

b. Show that the equilibrium point (P_o, Q_o) is not asymptotically stable.

III. Let

$$T = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \ ; \ x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1\}$$

and consider the system of differential equations

$$\dot{x}_1 = -y_1, \quad \dot{y}_1 = x_1, \quad \dot{x}_2 = -\sqrt{2} y_2, \quad \dot{y}_2 = \sqrt{2} x_2. \tag{1}$$

a. Show that if $t \mapsto (x_1(t), y_1(t), x_2(t), y_2(t))$ is a solution of the system of equations (1) such that $(x_1(t_o), y_1(t_o), x_2(t_o), y_2(t_o)) \in T$ then $(x_1(t), y_1(t), x_2(t), y_2(t)) \in T$ for all $t \in \mathbb{R}$.

b. Show that if

- $t \mapsto (x_1(t), y_1(t), x_2(t), y_2(t))$ is a solution of the system of equations (1) such that $(x_1(0), y_1(0), x_2(0), y_2(0)) \in T$,
- $P = (p_1, p_2, p_3, p_4) \in T$ and
- $\varepsilon > 0$

then there exists $t > 0$ such $(x_1(t)-p_1)^2+(y_1(t)-p_2)^2+(x_2(t)-p_3)^2+(y_2(t)-p_4)^2 < \varepsilon^2$.